General Relativity: Tutorial/ Discussion

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A. Central forces

In this section we will study the three-dimensional motion of a particle in a central force potential. Such a system obeys the equation of motion

$$m\ddot{x} = -\nabla V(r) \tag{1}$$

where the potential depends only on $r = |x|$.
Recall: 2-body problem in lab frame where masses m_1, m_2 exert force on each other such as:

$$\mathbf{F_{12}} = -G\frac{m_1\, m_2}{|\mathbf{r_1} - \mathbf{r_2}|^3}(\mathbf{r_1} - \mathbf{r_2}) \tag{2}$$

This is not obviously a central force problem — because: Each mass accelerates due to the other, the system as a whole can move so there's no single fixed center. Thus if we move to the centre of mass frame we get: $R_{CM} = \frac{m_1\mathbf{r_1}+m_2\mathbf{r_2}}{m_1+m_2}$ and reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$.
Then we split the full dynamics into two decoupled parts: (1) Motion of the COM (which is free if no external forces) and (2) Relative motion of $\mathbf{r}$ governed by:

$$\mu\ddot{\mathbf{r}} = -\frac{Gm_1m_2}{r^2}\hat{r} \tag{3}$$

This is a pure central force acting on a single particle of mass μ, the force is directed along $\mathbf{r}$.

Since both gravitational and electrostatic forces are of this form, solutions to this equation contain some of the most important results in classical physics. Our first line of attack in solving Eq.(1) is to use angular momentum. Recall that this is defined as,

$$\mathbf{L} = m\,\mathbf{x} \times \dot{\mathbf{x}} \tag{4}$$

It can be shown that angular momentum is conserved in a central potential. The proof is straightforward:

$$\frac{d\mathbf{L}}{dt} = m\,\mathbf{x} \times \ddot{\mathbf{x}} = -\mathbf{x} \times \nabla V = 0 \tag{5}$$

where the final equality follows because ∇V is parallel to $\mathbf{x}$.

The conservation of angular momentum has an important consequence: all motion takes place in a plane. This follows because L is a fixed, unchanging vector which, by construction, obeys, $\mathbf{L}.\mathbf{x} = 0$. So the position of the particle always lies in a plane perpendicular to $\mathbf{L}$. By the same argument, $\mathbf{L}.\dot{\mathbf{x}} = 0$, so the velocity of the particle also lies in the same plane. In this way the three dimensional dynamics is reduced to dynamics on a plane.

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B. Polar coordinates

We've learned that the motion lies in a plane. It will turn out to be much easier if we work with polar coordinates on the plane rather than Cartesian coordinates. For this reason, we take a brief detour to explain some relevant aspects of polar coordinates. To start, we rotate our coordinate system so that the angular momentum points in the z-direction and all motion takes place in the (x, y) plane. We then define the usual polar coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta \tag{6}$$

Our goal is to express both the velocity and acceleration in polar coordinates. We introduce two unit vectors, $\hat{r}$ and $\hat{\theta}$ in the direction of increasing r and θ respectively as shown in the diagram. Written in Cartesian form, these vectors

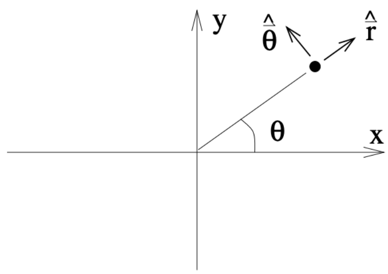

FIG. 1: Polar coordinates

are

$$\hat{r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \hat{\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{7}$$

These vectors form an orthornormal basis at every point on the plane. But the basis itself depends on which angle θ we sit at. Moving in the radial direction doesn't change the basis, but moving in the angular direction we have,

$$\frac{d\hat{r}}{d\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\theta}, \qquad \frac{d\hat{\theta}}{d\theta} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = -\hat{r} \tag{8}$$

This means that if the particle moves in a way such that θ changes with time, then the basis vectors themselves will also change with time. Let's see what this means for the velocity expressed in these polar coordinates. The position of a particle is written as, $\mathbf{x} = r\,\hat{r}$. From this we can compute the velocity, remembering that both r and the basis vector $\hat{r}$ can change with time. We get,

$$\dot{\mathbf{x}} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} \tag{9}$$

The second term in the above expression arises because the basis vectors change with time and is proportional to the angular velocity, $\dot{\theta}$ (Strictly speaking, this is the angular speed). Following similar procedure we obtain,

$$\ddot{\mathbf{x}} = \left(\ddot{\mathbf{r}} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta} \tag{10}$$

- Using polar coordinates derive explicitly the velocity and acceleration for the rotational motion as given above.
- Using above equations also find the velocity and acceleration for circular motion in terms of r, ω, where ω is the angular velocity.

C. Central force cont...

We've already seen that the three dimensional motion in a central force potential actually takes place in a plane. Let's write the equation of motion [Eq. 1] using the plane polar coordinates that we've just introduced. Since $V = V(r)$, the force itself can be written using,

$$m\left(\ddot{\mathbf{r}} - r\dot{\theta}^2\right)\hat{r} + m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta} = -\frac{dV}{dr}\,\hat{r} \tag{11}$$

- Using equation of motion show the angular momentum per unit mass (l) is also conserved in centra force field.
- Show that the magnitude of $\mathbf{L}$ comes out to be $|\mathbf{L}| = m\,l$

The $\hat{\theta}$ component of this is particularly simple. It is,

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \implies \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \tag{12}$$

A new conserved quantity emerges, $l = r^2\dot{\theta}$ does not change with time.

Magnitude of $\mathbf{L}$ becomes,

$$\mathbf{L} = m\mathbf{x}\times\dot{\mathbf{x}} = mr\hat{r}\times\left(\dot{r}\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right) = mr^2\dot{\theta}\left(\hat{r}\times\hat{\theta}\right) \tag{13}$$

Since $\hat{r}$ and $\hat{\theta}$ are orthogonal, unit vectors, $(\hat{r}\times\hat{\theta})$ is also a unit vector. The magnitude of the angular momentum vector is therefore $|L| = m\,l$.

Let's now look at the $\hat{r}$ component of the equation of motion Eq. (11). It is,

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{dV}{dr} \tag{14}$$

Using the fact that $l = r^2\dot{\theta}$ is conserved, we can write this as,

$$m\,\ddot{r} = -\frac{dV}{dr} + \frac{m\,l^2}{r^3} \tag{15}$$

Note: We started from Eq. (1) with a complicated, three dimensional problem. We used the direction of the angular momentum to reduce it to a two dimensional problem, and the magnitude of the angular momentum to reduce it to a one dimensional problem. This was all possible because angular momentum is conserved. This gives us an idea that how important conserved quantities are when it comes to solving any systems. Now one can rewrite the Eq. (15) as,

$$m\,\ddot{r} = -\frac{dV_{\text{eff}}}{dr} \qquad \text{where,} \quad V_{\text{eff}}(r) = V(r) + \frac{ml^2}{2r^2} \tag{16}$$

The extra term, $\frac{ml^2}{2r^2}$ is called the angular momentum barrier. It stops the particle getting too close to the origin.

D. The effective potential

One can check that the effective potential can indeed be thought of as part of the energy of the full system.

- Verify that the energy in 3-dimensions can be written as, $E = \frac{1}{2}m\,\dot{r}^2 + V_{\text{eff}}(r)$. (note: it reduced to the energy of the effective one dimensional system). The effective potential energy is the real potential energy, together with a contribution from the angular kinetic energy.
- Starting with the most useful central potential $V(r) = -k/r$ for $k > 0$, the effective potential becomes $V_{\text{eff}} = -k/r + \frac{ml^2}{2r^2}$. Considering V_{eff}, show that $E_{\text{min}} = -k^2/2ml^2$.

- Explain physical implication for the particle having minimum energy in this potential.
- What would you expect for the radial distance if you increase the angular momentum of the particle?
- What you can infer for V_{eff} and r if you put $l = 0$

- Consider a general potential $V(r)$. Can we derive that when do circular orbits exist? And when are they stable? Hint: Circular orbits exist whenever there exists a solution with $l \neq 0$ and $\dot{r} = 0$ for all time.
- in a Universe with d spatial dimensions, the law of gravity would be $F \sim 1/r^{d-1}$ corresponding to a potential energy $V \sim 1/r^{d-2}$. We see that circular planetary orbits are only stable in $d < 4$ spatial dimensions.

Returning to the case of general V_{eff}, if we want to understand how the radial position $r(t)$ changes with time, then the problem is essentially solved. Since the energy E is conserved, we have,

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) \tag{17}$$

which we can view as a first order differential equation for dr/dt. Integrating we get,

$$t = \pm\sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E - V_{\text{eff}}(r)}} \tag{18}$$

However, except for a few very special choices of V_{eff}, the integral is kind of complicated. Thus we can think of that often trying to figure out $r(t)$ is not necessarily the information that we are looking for. It is better to take a more generalised/ global approach, and try to understand the whole trajectory of the particle, rather than its position at any given time. Mathematically, this means that we will try to understand something about the shape of the orbit. In order to do that we use a reparametrisation such as $u = \frac{1}{r}$.

- Using the above reparametrisation verify that the Eq. (15) becomes:

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}F(1/u) \tag{19}$$

You can replace $-dV/dr = F(r)$ and then derive the above.

E. The Kepler Problem

The Kepler problem is the name given to understanding planetary orbits about a star. It is named after the astronomer Johannes Kepler. Now it is well known that the inverse square force law of gravitation is described by the central potential $V(r) = -km/r$, where $k = GM$.
For this potential the orbit equation i.e. Eq. (19) becomes,

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{l^2} \tag{20}$$

This is just the equation for a harmonic oscillator with its centre displaced by k/l^2. We can write the most general solution as,

$$u = A\cos(\theta - \theta_0) + \frac{k}{l^2} \tag{21}$$

with A and θ_0 are integration constants. At the point where the orbit is closest to the origin (the periapsis), u is largest. From our solution, we have $u_{\max} = A + k/l^2$. We will choose to orient our polar coordinates so that the periapsis occurs at $\theta = 0$. This choice means that set $\theta_0 = 0$. In terms of our original variable $r = 1/u$, we have the final expression for the orbit:

$$r = \frac{r_0}{e\,\cos\theta + 1} \qquad \text{where,} \qquad r_0 = \frac{l^2}{k} \quad \text{and} \qquad e = \frac{Al^2}{k} \tag{22}$$

One can perceive that for $\theta = 0$ its a perihelion point and for $\theta = \pi$ its aphelion point and for lets say $\theta = 2\pi$ its again the same value for $r = \frac{r_0}{e+1}$, which states that particle returns exactly to the same perihelion point.

F. Orbital precession

- For extremely massive objects, Newton's theory of gravity needs a replacement and we all know the answer to this problem i.e. the solution is Einstein's theory of general relativity which describes how gravity can be understood as the curvature of spacetime. Einstein GR brought a new correction term to the effective potential which we derived earlier.

- Let us address this question in the upcoming discussion. To find this we consider Schwarzschild geometry and a test particle moving under the Sch. spacetime. Thus, we consider a test particle moving in the Schwarzschild geometry, described by the line element:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{23}$$

Note we have kept $c = 1$. Orient the coordinate system so that the radial projection of the orbit coincides with the equator, $\theta = \pi/2$, of the polar coordinates. In GR if K^μ is a Killing vector we know that,

$$K^\mu \frac{dx^\mu}{d\lambda} = \text{constant} \tag{24}$$

In addition, we always have another constant of the motion for geodesics: the geodesic equation (together with metric compatibility) implies that the quantity

$$\epsilon = -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{25}$$

is constant along the path. (For any trajectory we can choose the parameter λ such that ϵ is a constant. Of course, for a massive particle we typically choose $\lambda = \tau$, and this relation simply becomes $\epsilon = -g_{\mu\nu}U^\mu U^\nu = +1$. For massless particles, which move along null trajectories, we always have $\epsilon = 0$, and this equation does not fix the parameter λ. For the spacelike geodesic we choose $\epsilon = -1$.
Note that the line element in Eq. (23), exhibits that the geometry is unaffected by the translations $t \to t + \Delta t$, $\phi \to \phi + \Delta\phi$. Thus in Sch spacetime we get the Killing vectors associated with the coordinates t, ϕ. The Killing vector can be written as,

$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0)$$

$$K_\mu = \left(-\left(1 - \frac{2GM}{r}\right), 0, 0, 0\right) \tag{26}$$

Similarly,

$$R^\mu = (\partial_\phi)^\mu = (0, 0, 0, 1)$$

$$R_\mu = \left(0, 0, 0, r^2 \sin^2\theta\right) \tag{27}$$

Using $\theta = \pi/2$, verify that the conserved quantities become the energy and the angular momentum of the particle as below:

$$E = -K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\lambda} \qquad L = R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \tag{28}$$

For massless particles, these can be thought of as the conserved energy and angular momentum, while for massive particles they are the conserved energy and angular momentum per unit mass of the particle.
This Eq. (25) becomes, (Please verify)

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -\epsilon \tag{29}$$

Further this reduces to, (Please verify)

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0 \tag{30}$$

Please check the above equation further reduces to,

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E} \tag{31}$$

Here,

$$V(r) = \frac{\epsilon}{2} - \frac{\epsilon GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \qquad \mathcal{E} = \frac{E^2}{2} \tag{32}$$

In Eq. (31) we have precisely the equation for a classical particle of unit mass and "energy" $\mathcal{E}$ moving in one-dimensional potential given by $V(r)$. Now using $\epsilon = 1$ for massive particle one can write Eq. (31) follows Please verify the following equation,

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2} - \frac{2GM}{L^2}r^3 + r^2 - 2GMr = \frac{2\mathcal{E}}{L^2}r^4 \tag{33}$$

To solve the above equation take $x = \frac{L^2}{GMr}$, which reduces Eq. (33) to,

$$\left(\frac{dx}{d\phi}\right)^2 + \frac{L^2}{G^2M^2} - 2x + x^2 - \frac{2G^2M^2}{L^2}x^3 = \frac{2\mathcal{E}L^2}{G^2M^2} \tag{34}$$

Differentiating this with respect ot ϕ, one obtains,

$$\frac{d^2x}{d\phi^2} - 1 + x = \frac{3G^2M^2}{L^2}x^2 \tag{35}$$

In a Newtonian calculation, the last term would be absent, and we could solve for x exactly; here, we can treat it as a perturbation. We expand x into Newtonian solution plus a small deviation, $x = x_0 + x_1$.
Zeroth order part of Eq. (35) becomes,

$$\frac{d^2x_0}{d\phi^2} - 1 + x_0 = 0 \tag{36}$$

and the first-order part is:

$$\frac{d^2x_1}{d\phi^2} + x_1 = \frac{3G^2M^2}{L^2}x_0^2 \tag{37}$$

The solution for the zeroth-order equation can be written:

$$x_0 = 1 + e\,\cos\phi \tag{38}$$

This is the standard result of Newton or Kepler; to describes a perfect ellipse, with e the eccentricity. An ellipse is specified by the semi-major axis a, the distance from the center to the farthest point on the ellipse, and the semi-minor axis b, the distance from the center to the closest point. The eccentricity satisfies $e^2 = 1 - b^2/a^2$. Plugging the Newtonian solution into the first-order equation, we obtain:

$$\frac{d^2x_1}{d\phi^2} + x_1 = \frac{3G^2M^2}{L^2}(1 + e\cos\phi)^2 \tag{39}$$

Solving this equation for x_1 one gets, (Please verify), Ref: Sean Carroll book

$$x_1 = \frac{3G^2M^2}{L^2}\left[\left(1 + \frac{e^2}{2}\right) + e\phi\sin\phi - \frac{e^2}{6}\cos 2\phi\right] \tag{40}$$

Furthermore it gives,

$$x = 1 + e\cos\phi + \frac{3G^2M^2\,e}{L^2}\phi\sin\phi \tag{41}$$

This can be written as,

$$x = 1 + e\cos[(1-\alpha)\phi] \qquad \alpha = \frac{3G^2M^2}{L^2} \tag{42}$$

$$\implies r = \frac{L^2}{GM\,[1 + e\cos[(1-\alpha)\phi]]} \tag{43}$$

The equivalence of Eqs. (41) and (42) can be seen by expanding $\cos[(1-\alpha)\phi]$ as a power series in the small parameter α:

$$\cos[(1-\alpha)\phi] = \cos\phi + \alpha\frac{d}{d\alpha}\cos[(1-\alpha)\phi]_{\alpha=0} = \cos\phi + \alpha\phi\sin\phi \tag{44}$$

We have therefore found that, during each orbit of the planet, perihelion advances by an angle: $\Delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2}$ show how $2\pi\alpha$ comes . Now to replace the angular mom. we write that ordinary ellipse satisfies:

$$r = \frac{(1-e^2)\,a}{1 + e\cos\phi} \tag{45}$$

where a is the semi-major axis. Comparing to our zeroth-order solution (38) and the definition of $x = \frac{L^2}{GMr}$, we see that, $L^2 \approx GM(1-e^2)\,a$ and we get,

$$\Delta\phi = \frac{6\pi GM}{c^2(1-e^2)a} \tag{46}$$

Historically, the precession of Mercury was the first test of GR. In fact it was known before Einstein invented GR that there was an apparent discrepancy in Mercury's orbit, and a number of solutions had been proposed. For the motion of Mercury around the Sun, the relevant orbital parameters are,

$$\frac{GM_{\rm sun}}{c^2} = 1.48\times10^5\,{\rm cm} \qquad a = 5.79\times10^{12}\,{\rm cm} \qquad e = 0.2056 \qquad c = 3\times10^{10}\,{\rm cm/sec} \tag{47}$$

This gives,

$$\Delta\phi_{\rm mercury} = 5.01\times10^{-7}{\rm radians/orbit} = 43.0''/{\rm century} \tag{48}$$

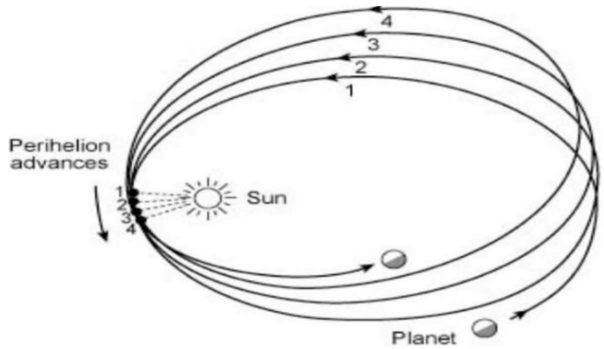

FIG. 2: Perihelion precession